Chapter 1

Introduction

1.1 Purpose of Course

Course covers the theory of computers:

• Not concerned with actual hardware and software.

• More interested in abstract questions of the frontiers of capability of computers.

• More specifically, what can and what cannot be done by any existing computer or any computer ever built in the future.

• We will study different types of theoretical machines that are mathematical models for actual physical processes.

• By considering the possible inputs on which these machines can work, we can analyze their various strengths and weaknesses.

• We can then develop what we may believe to be the most powerful machine possible.

• Surprisingly, it will not be able to perform every task, even some easily described tasks.
1.2 Mathematical Background

- In this class, we will be seeing a number of theorems and proofs.
- To be able to understand how to prove a theorem, we first have to understand how theorems are stated.
- Many (but not all) theorems are stated as “if $p$, then $q$”, where $p$ and $q$ are statements.

**Example:** If a word $w$ has more $e$’s than $o$’s, then $w$ has at least one $e$.

**Example:** If a word $w$ has $m$ a’s and $n$ e’s in it, then the word $w$ has at least $m + n$ letters in all.

**Example:** If $x^2 = 0$, then $x = 0$.

So what does “if $p$, then $q$” mean?

- If a theorem stated in this form is to be true, then it means that if $p$ is true, then $q$ must also be true.
- Note that this does not say that if $q$ is true, then $p$ must also be true. This may or may not be the case.

**Example:** The statement, “If a word $w$ has at least one $e$, then $w$ has more $e$’s than $o$’s” is not true.

For example, consider the word “exploration” or “Exxon.”

**Example:** If a word $w$ has at least $m + n$ letters in all, then the word $w$ has $m$ a’s and $n$ e’s in it.

For example, suppose $m = n = 1$, and consider the word “goof.”

**Example:** If $x = 0$, then $x^2 = 0$.

So now how do we prove a result?

We do it by arguing very carefully, where each step in our argument follows logically from the previous step.

There are several ways of proving that a statement “if $p$, then $q$” holds:
• One way is to use a direct argument:

**Example:** Prove: If a word \( w \) has more \( e \)'s than \( o \)'s, then \( w \) has at least one \( e \).

**Proof.** Let \( n_e \) be the number of \( e \)'s in \( w \), and let \( n_o \) be the number of \( o \)'s in \( w \). Since \( w \) has more \( e \)'s than \( o \)'s, we must have that \( n_e > n_o \), or in other words \( n_e \geq n_o + 1 \). But since \( w \) cannot have fewer than zero \( o \)'s, we must have that \( n_o \geq 0 \). Therefore, \( n_e \geq n_o + 1 \geq 0 + 1 = 1 \). Thus, \( w \) has at least one \( e \). ■

• Another way of proving results is by contradiction. We do this by assuming that \( p \) is true and that \( q \) is not true, and then showing that an inconsistency results.

**Example:** Prove: If \( x^2 = 0 \), then \( x = 0 \).

**Proof.** Suppose that \( x^2 = 0 \) but \( x \neq 0 \). Then either \( x > 0 \) or \( x < 0 \). But if \( x > 0 \), then \( x^2 > 0 \), and if \( x < 0 \), then \( x^2 > 0 \). In either case, \( x^2 > 0 \). This contradicts the assumption that \( x^2 = 0 \). ■

**Example:** Prove: If \( x > 0 \) with \( x \in \mathbb{R} \), then \( x^2 > 0 \).

**Proof.** Suppose that \( x^2 = 0 \). Then \( x = 0 \) so \( x \neq 0 \). ■

There are several equivalent ways of stating “if \( p \), then \( q \)”

• “if not \( q \), then not \( p \)”
• “\( p \) only if \( q \)”
• “\( q \) if \( p \)”
• “\( p \) implies \( q \)”
• “\( p \) is sufficient for \( q \)”
• “\( q \) is necessary for \( p \)”

**Example:** Let \( x \) be a real number. If \( x > 0 \), then \( x^2 > 0 \).

This is equivalent to stating
• “If $x^2 > 0$ is not true (i.e., $x^2 \leq 0$), then $x > 0$ is not true (i.e., $x \leq 0$).”
• This is also equivalent to stating “$x > 0$ only if $x^2 > 0$.”
• This is also equivalent to stating “$x^2 > 0$ if $x > 0$.”
• This is also equivalent to stating “$x > 0$ implies $x^2 > 0$.”

Often, the two statements

1. “$p$ only if $q$” (i.e., “if $p$, then $q$”) and
2. “$p$ if $q$” (i.e., “if $q$, then $p$”)

are combined into “$p$ if and only if $q$” (or “$p$ is a necessary and sufficient condition for $q$”).

In order for this statement to be true, we need to show that both statements 1 and 2 above are true.

**Definition:** An integer $n$ is an **even number** if $n = 2k$ for some $k = 0, 1, 2, 3, \ldots$.

**Definition:** An integer $n$ is an **odd number** if $n = 2k + 1$ for some $k = 0, 1, 2, 3, \ldots$.

**Definition:** An integer $n$ is a **positive even number** if $n = 2k$ for some $k = 1, 2, 3, \ldots$. 