Chapter 7

Kleene's Theorem

7.1 Kleene's Theorem

The following theorem is the most important and fundamental result in the theory of FA's:

Theorem 6 *Any language that can be defined by either*

- *regular expression, or*
- *finite automata, or*
- *transition graph*

can be defined by all three methods.

Proof. The proof has three parts:

Part 1: (FA $\Rightarrow$ TG) Every language that can be defined by an *FA* can also be defined by a *transition graph*.

Part 2: (TG $\Rightarrow$ RegExp) Every language that can be defined by a *transition graph* can also be defined by a *regular expression*.

Part 3: (RegExp $\Rightarrow$ FA) Every language that can be defined by a *regular expression* can also be defined by an *FA*.

7.2 Proof of Part 1: FA ⇒ TG

- We previously saw that every FA is also a transition graph.
- Hence, any language that has been defined by a FA can also be defined by a transition graph.

7.3 Proof of Part 2: TG ⇒ RegExp

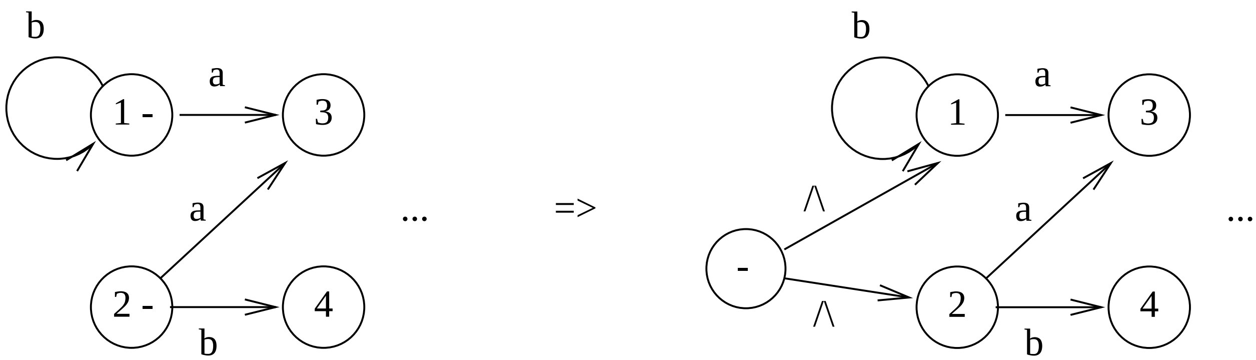
- We will give a constructive algorithm for proving part 2.
- Thus, we will describe an algorithm to take any transition graph T and form a regular expression corresponding to it.
- The algorithm will work for any transition graph T.
- The algorithm will finish in finite time.

An overview of the algorithm is as follows:

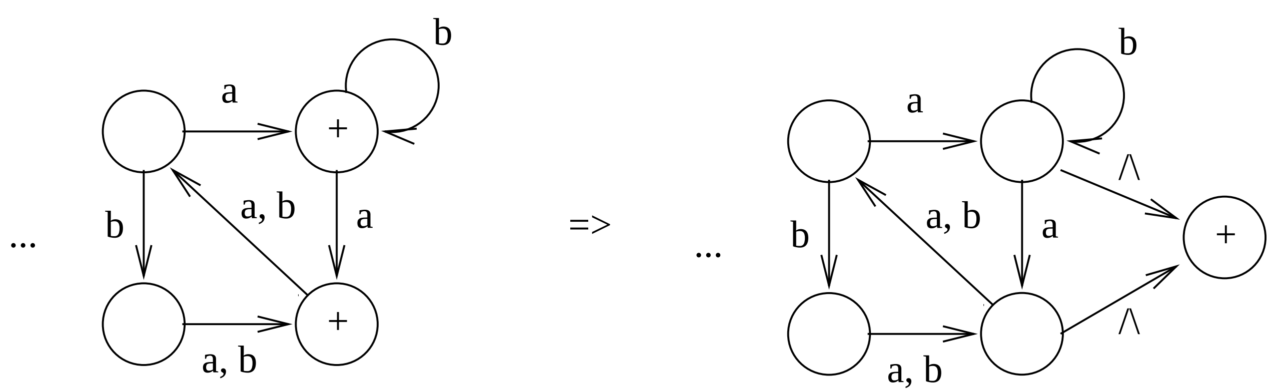
- Start with any transition graph T.
- First, transform it into an equivalent transition graph having only one start state and one final state.
- In each following step, eliminate either some states or some arcs by transforming the TG into another equivalent one.
- We do this by replacing the strings labelling arcs with regular expressions.
- We can traverse an arc labelled with a regular expression using any string that can be generated by the regular expression.
- End up with a TG having only two states, start and final, and one arc going from start to final.
- The final TG will have a regular expression on its one arc
- Note that in each step we eliminate some states or arcs.
- Since the original TG has a finite number of states and arcs, the algorithm will terminate in a finite number of iterations.

Algorithm:

1. If T has more than one start state, add a new state and add arcs labeled Λ going to each of the original start states.

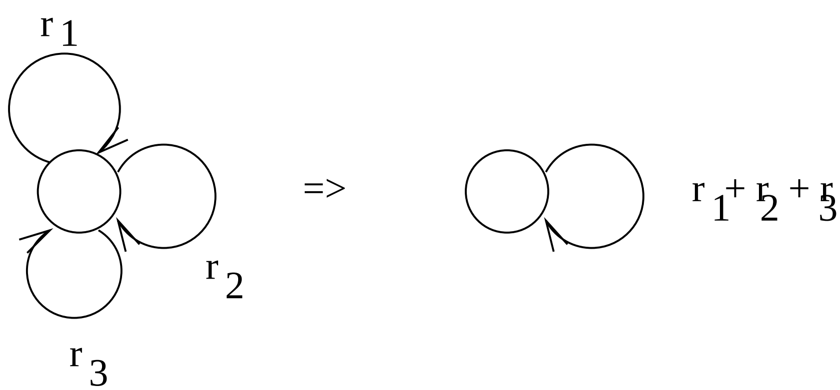


2. If T has more than one final state, add a new state and add arcs labeled Λ going from each of the original final states to the new state. Need to make sure the final state is different than the start state.

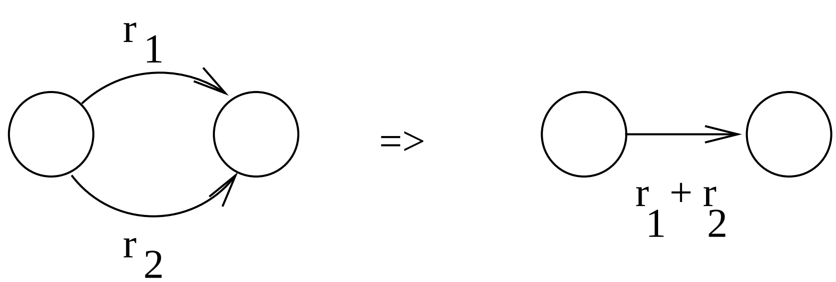


3. Now we give an iterative procedure for eliminating states and arcs

   (a) If T has some state with $n > 1$ loops circling back to itself, where the loops are labeled with regular expressions $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, then replace the n loops with a single loop labeled with the regular expression $\mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_n$.



   (b) If two states are connected by $n > 1$ direct arcs in the same direction, where the arcs are labelled with the regular expressions $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, then replace the n arcs with a single arc labeled with the regular expression $\mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_n$.

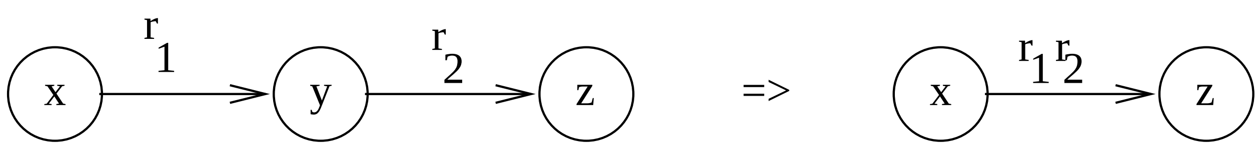


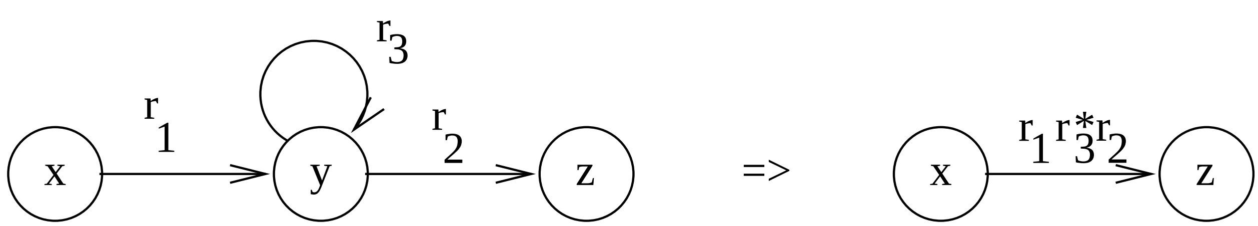
(c) *Bypass operation*:

i. If there are three states x, y, z such that

- there is an arc from x to y labelled with the regular expression $\mathbf{r}_1$ and
- an arc from y to z labelled with the regular expression $\mathbf{r}_2$,

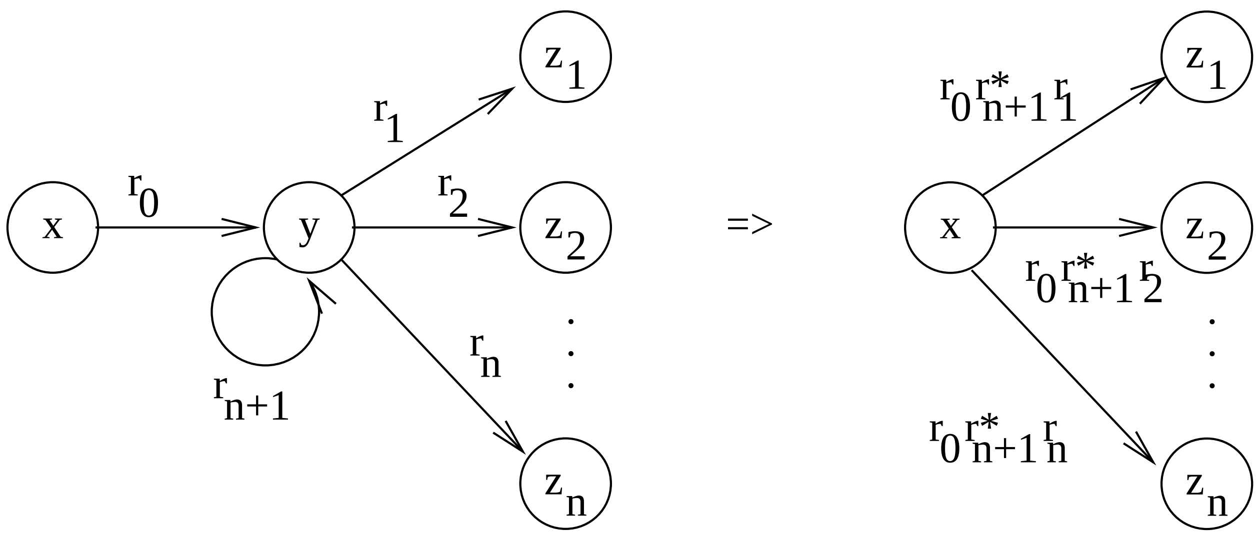
then replace the two arcs and the state y with a single arc from x to z labelled with the regular expression $\mathbf{r}_1\mathbf{r}_2$.





ii. If there are
   - $n+2$ states $x, y, z_1, z_2, \ldots, z_n$ such that there is an arc from x to y labelled with the regular expression $\mathbf{r}_0$, and
   - an arc from y to z_i, $i = 1, 2, \ldots, n$, labelled with the regular expression $\mathbf{r}_i$, and
   - an arc from y back to itself labelled with regular expression $\mathbf{r}_{n+1}$,

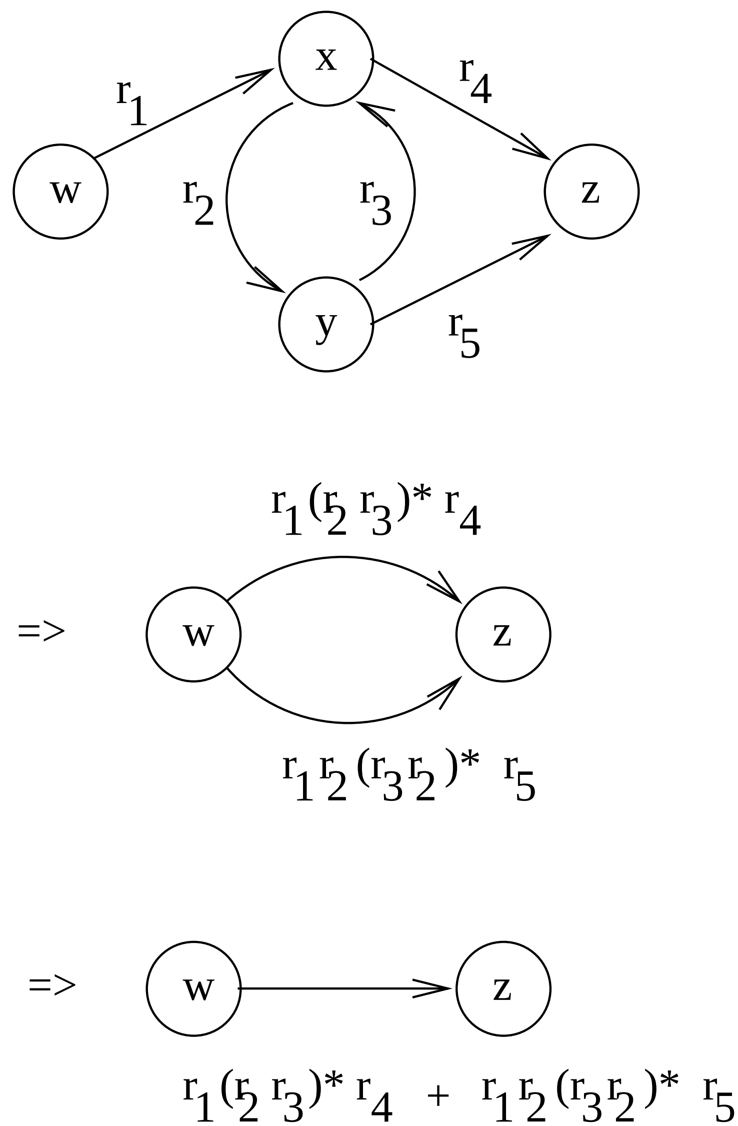
   then replace the $n+1$ original arcs and the state y with n arcs from x to z_i, $i = 1, 2, \ldots, n$, each labelled with the regular expression $\mathbf{r}_0\mathbf{r}_{n+1}\mathbf{r}_i$.



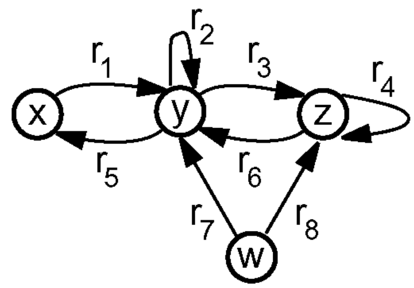
iii. If any other arcs led directly to y, divert them directly to the z_i's.

iv. Need to make sure that all paths possible in the original TG are still possible after the bypass operation.

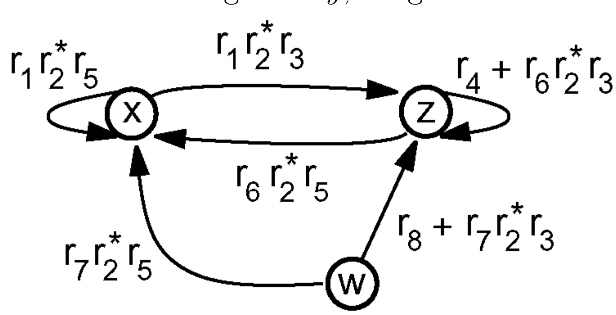
- Example



- Example:

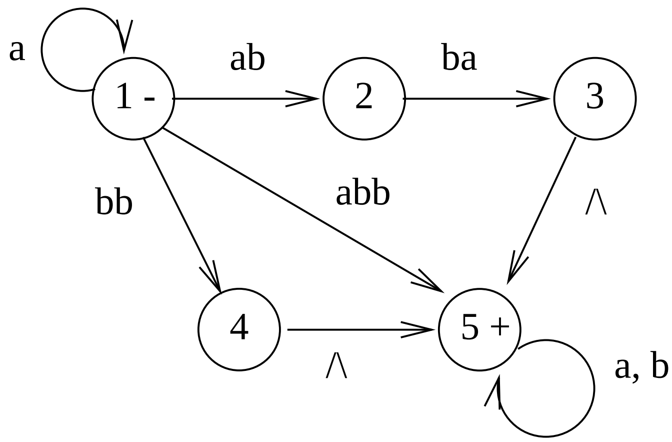


  - Suppose we want to get rid of state y.
  - Need to account for all paths that go through state y.
  - There are arcs coming from x, w, and z going into y.
  - There are arcs from y to x and z.
  - Thus, we need to account for each possible path from a state having an arc into y (i.e., x, w, z) to each state having an arc from y (i.e., x, z)
  - Thus, we need to account for the paths from
    - x to y to x, which has regular expression $\mathbf{r}_1\mathbf{r}_2^*\mathbf{r}_5$
    - x to y to z, which has regular expression $\mathbf{r}_1\mathbf{r}_2^*\mathbf{r}_3$
    - w to y to x, which has regular expression $\mathbf{r}_7\mathbf{r}_2^*\mathbf{r}_5$
    - w to y to z, which has regular expression $\mathbf{r}_7\mathbf{r}_2^*\mathbf{r}_3$
    - z to y to x, which has regular expression $\mathbf{r}_6\mathbf{r}_2^*\mathbf{r}_5$
    - z to y to z, which has regular expression $\mathbf{r}_6\mathbf{r}_2^*\mathbf{r}_3$
  - Thus, after eliminating state y, we get the following:



v. Never delete the unique start or final state.

Example:



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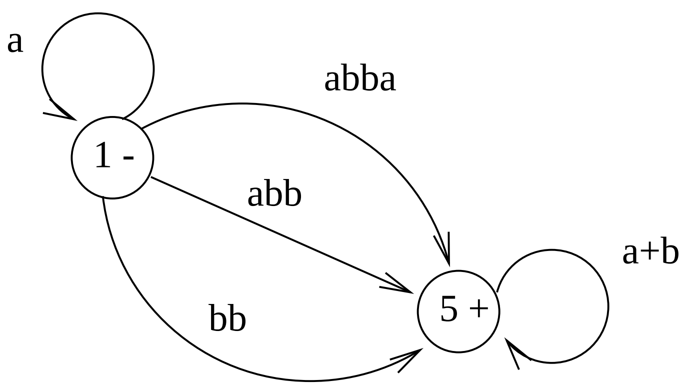

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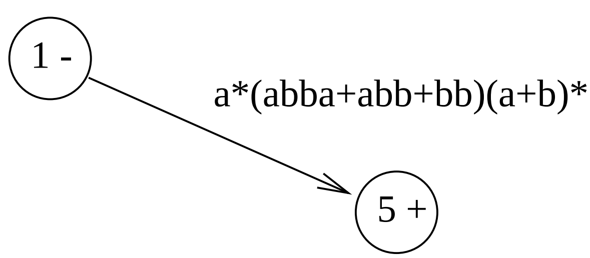
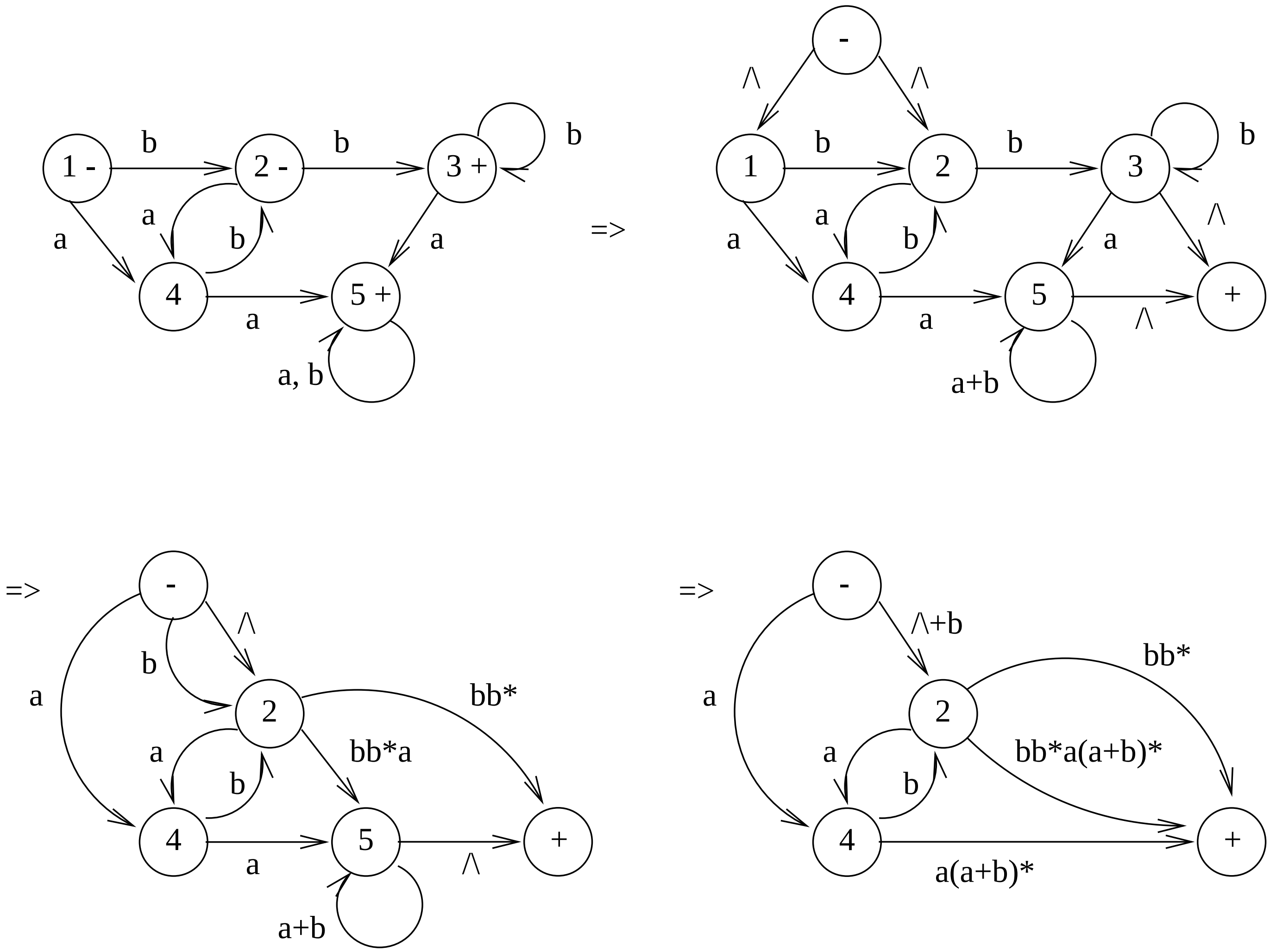


Example: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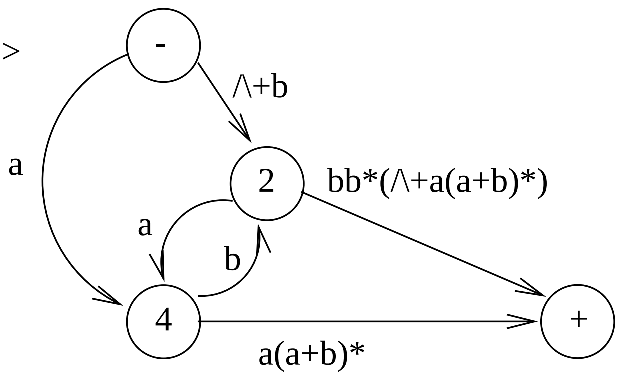

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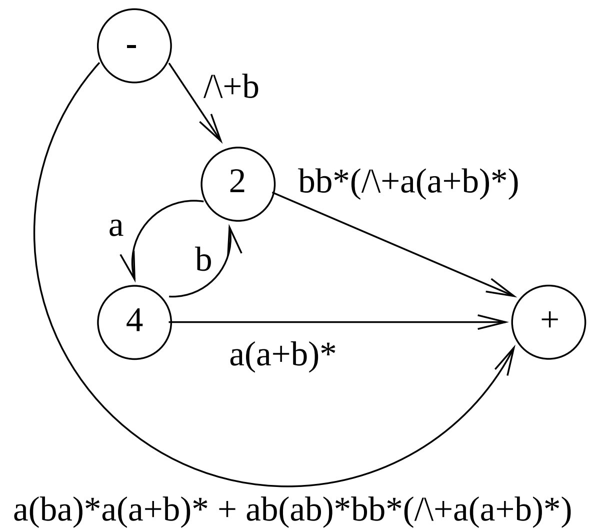

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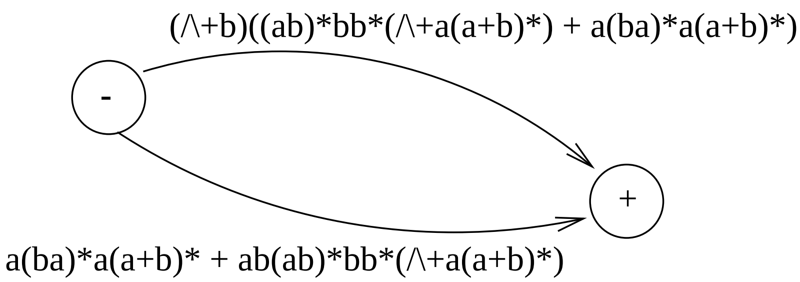


=> (/\+b)((ab)*bb*(/\+a(a+b)*) + a(ba)*a(a+b)*) + a(ba)*a(a+b)* + ab(ab)*bb*(/\+a(a+b)*)

7.4 Proof of Part 3: RegExp ⇒ FA

To show: every language that can be defined by a *regular expression* can also be defined by a *FA*.

We will do this by using a *recursive definition* and a constructive algorithm.

Recall

- every regular expression can be built up from the letters of the alphabet and Λ and $\emptyset$.
- Also, given some existing regular expressions, we can build new regular expressions by applying the following operations:
  1. union (+)
  2. concatenation
  3. closure (Kleene star)
- We will not include r^+ in our discussion here, but this will not be a problem since $r^+ = rr^*$.

Recall that we had the following recursive definition for regular expressions:

Rule 1: If $x \in \Sigma$, then $\mathbf{x}$ is a regular expression. $\boldsymbol{\Lambda}$ is a regular expression. $\emptyset$ is a regular expression.

Rule 2: If $\mathbf{r}_1$ and $\mathbf{r}_2$ are regular expressions, then $\mathbf{r}_1 + \mathbf{r}_2$ is a regular expression.

Rule 3: If $\mathbf{r}_1$ and $\mathbf{r}_2$ are regular expressions, then $\mathbf{r}_1\mathbf{r}_2$ is a regular expression.

Rule 4: If $\mathbf{r}_1$ is a regular expression, then $\mathbf{r}_1^*$ is a regular expression.

Based on the above recursive definition for regular expressions, we have the following recursive definition for FA's associated with regular expressions:

Rule 1:

- There is an FA that accepts the language L defined by the regular expression $\mathbf{x}$; i.e., $L = \{x\}$, where $x \in \Sigma$, so language L consists of only a single word and that word is the single letter x.
- There is an FA that accepts the language defined by regular expression $\boldsymbol{\Lambda}$; i.e., the language $\{\Lambda\}$.
- There is an FA defined by the regular expression $\emptyset$; i.e., the language with no words, which is $\emptyset$.

Rule 2: If there is an FA called FA_1 that accepts the language defined by the regular expression $\mathbf{r}_1$ and there is an FA called FA_2 that accepts the language defined by the regular expression $\mathbf{r}_2$, then there is an FA called FA_3 that accepts the language defined by the regular expression $\mathbf{r}_1 + \mathbf{r}_2$.

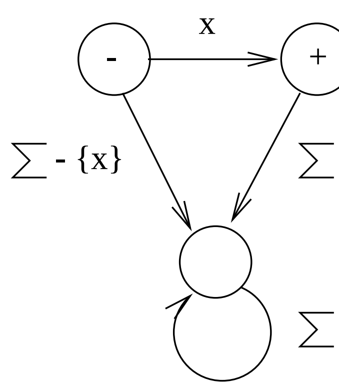
Rule 3: If there is an FA called FA_1 that accepts the language defined by the regular expression $\mathbf{r}_1$ and there is an FA called FA_2 that accepts the language defined by the regular expression $\mathbf{r}_2$, then there is an FA called FA_3 that accepts the language defined by the regular expression $\mathbf{r}_1\mathbf{r}_2$, which is the concatenation.

Rule 4: If there is an FA called FA_1 that accepts the language defined by the regular expression $\mathbf{r}_1$, then there is an FA called FA_2 that accepts the language defined by the regular expression $\mathbf{r}_1^*$.

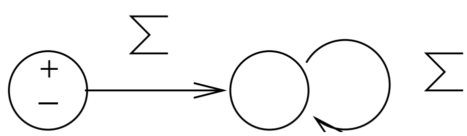
Let's now show that each of the rules hold by construction:

Rule 1: There is an FA that accepts the language L defined by the regular expression $\mathbf{x}$; i.e., $L = \{x\}$, where $x \in \Sigma$. There is an FA that accepts language defined by the regular expression $\mathbf{\Lambda}$. There is an FA that accepts the language defined by the regular expression $\emptyset$.

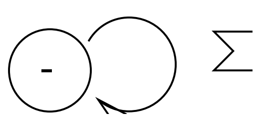
- If $x \in \Sigma$, then the following FA accepts the language $\{x\}$:



- An FA that accepts the language $\{\Lambda\}$ is



- An FA that accepts the language $\emptyset$ is



Rule 2: If there is an FA called FA_1 that accepts the language defined by the regular expression $\mathbf{r}_1$ and there is an FA called FA_2 that accepts the language defined by the regular expression $\mathbf{r}_2$, then there is an FA called FA_3 that accepts the language defined by the regular expression $\mathbf{r}_1 + \mathbf{r}_2$.

- Suppose regular expressions $\mathbf{r}_1$ and $\mathbf{r}_2$ are defined with respect to a common alphabet Σ.
- Let L_1 be the language generated by regular expression $\mathbf{r}_1$.
- L_1 has finite automaton FA_1.
- Let L_2 be the language generated by regular expression $\mathbf{r}_2$.
- L_2 has finite automaton FA_2.
- Regular expression $\mathbf{r}_1 + \mathbf{r}_2$ generates the language $L_1 + L_2$.
- Recall $L_1 + L_2 = \{w \in \Sigma^* : w \in L_1 \text{ or } w \in L_2\}$.
- Thus, $w \in L_1 + L_2$ if and only if w is accepted by either FA_1 or FA_2 (or both).
- We need FA_3 to accept a string if the string is accepted by FA_1 or FA_2 or both.
- We do this by constructing a new machine FA_3 that simultaneously keeps track of where the input would be if it were running on FA_1 and where the input would be if it were running on FA_2.
- Suppose FA_1 has states $x_1, x_2, \ldots, x_m$, and FA_2 has states $y_1, y_2, \ldots, y_n$.
- Assume that x_1 is the start state of FA_1 and that y_1 is the start state of FA_2.
- We will create FA_3 with states of the form (x_i, y_j).
- The number of states in FA_3 is at most mn, where m is the number of states in FA_1 and n is the number of states in FA_2.
- Each state in FA_3 corresponds to a state in FA_1 and a state in FA_2.
- FA_3 accepts string w if and only if either FA_1 or FA_2 accepts w.
- So final states of FA_3 are those states (x, y) such that x is a final state of FA_1 or y is a final state of FA_2.

We use the following algorithm to construct FA_3 from FA_1 and FA_2.

- Suppose that Σ is the alphabet for both FA_1 and FA_2.
- Given $FA_1 = (K_1, \Sigma, \pi_1, s_1, F_1)$ with
  - Set of states $K_1 = \{x_1, x_2, \ldots, x_m\}$
  - $s_1 = x_1$ is the initial state
  - $F_1 \subset K_1$ is the set of final states of FA_1.
  - $\pi_1 : K_1 \times \Sigma \rightarrow K_1$ is the transition function for FA_1.
- Given $FA_2 = (K_2, \Sigma, \pi_2, s_2, F_2)$ with
  - Set of states $K_2 = \{y_1, y_2, \ldots, y_n\}$
  - $s_2 = y_1$ is the initial state
  - $F_2 \subset K_2$ is the set of final states of FA_2.
  - $\pi_2 : K_2 \times \Sigma \rightarrow K_2$ is the transition function for FA_2.
- We then define $FA_3 = (K_3, \Sigma, \pi_3, s_3, F_3)$ with
  - Set of states $K_3 = K_1 \times K_2 = \{(x, y) : x \in K_1, y \in K_2\}$
  - The alphabet of FA_3 is Σ.
  - FA_3 has transition function $\pi_3 : K_3 \times \Sigma \rightarrow K_3$ with

$$\pi_3((x, y), \ell) = (\pi_1(x, \ell), \pi_2(y, \ell)).$$

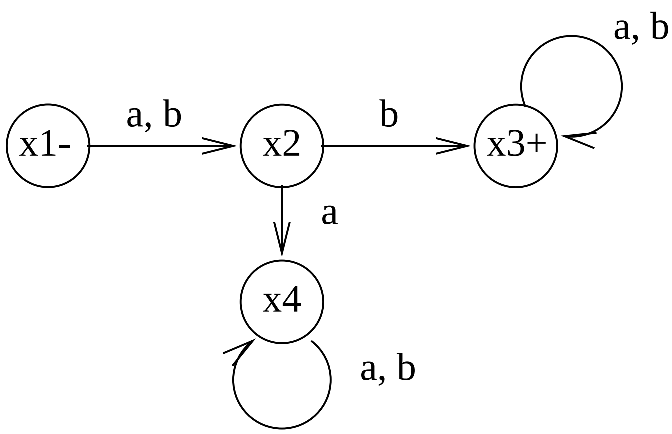
  - The initial state $s_3 = (s_1, s_2)$.
  - The set of final states

$$F_3 = \{(x, y) \in K_1 \times K_2 : x \in F_1 \text{ or } y \in F_2\}.$$

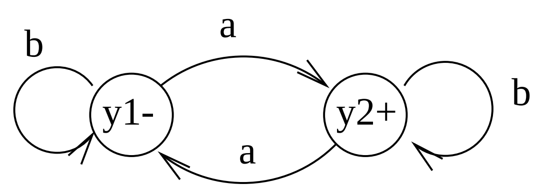
- Since $K_3 = K_1 \times K_2$, the number of states in the new machine FA_3 is $|K_3| = |K_1| \cdot |K_2|$.
  - But we can leave out a state $(x, y) \in K_1 \times K_2$ from K_3 if (x, y) is not reachable from FA_3's initial state (s_1, s_2).
  - This would result in fewer states in K_3, but still we have $|K_1| \cdot |K_2|$ as an upper bound for $|K_3|$; i.e., $|K_3| \leq |K_1| \cdot |K_2|$.

Example: $L_1 = \{$ words with b as second letter$\}$
with regular expression $\mathbf{r}_1 = (\mathbf{a}+\mathbf{b})\mathbf{b}(\mathbf{a}+\mathbf{b})^*$
$L_2 = \{$ words with odd number of a's$\}$
with regular expression $\mathbf{r}_2 = \mathbf{b}^*\mathbf{a}(\mathbf{b}+\mathbf{ab}^*\mathbf{a})^*$

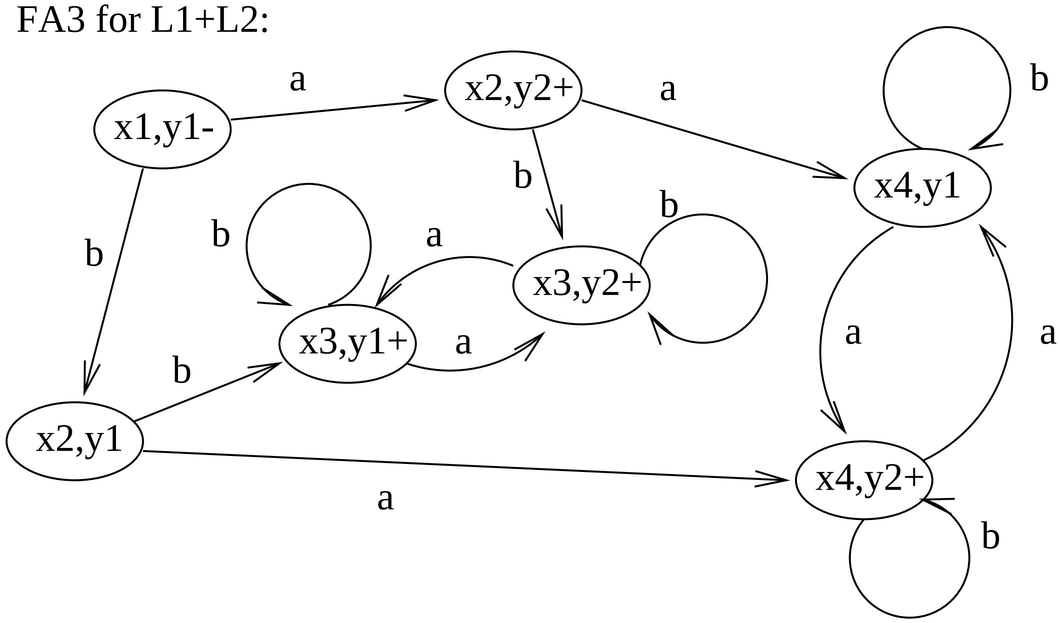
FA1 for L1:



FA2 for L2:



FA3 for L1+L2:



Rule 3: If there is an FA called FA_1 that accepts the language defined by the regular expression $\mathbf{r}_1$ and there is an FA called FA_2 that accepts the language defined by the regular expression $\mathbf{r}_2$, then there is an FA called FA_3 that accepts the language defined by the regular expression $\mathbf{r}_1\mathbf{r}_2$.

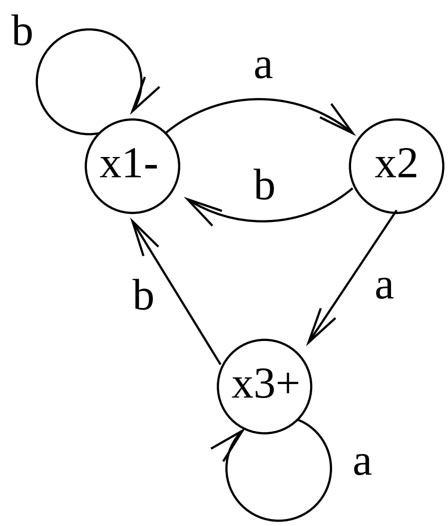
For this part,

- we need FA_3 to accept a string if the string can be factored into two substrings, where the first factor is accepted by FA_1 and the second factor is accepted by FA_2.
- One problem is we don't know when we reach the end of the first factor and the beginning of the second factor.

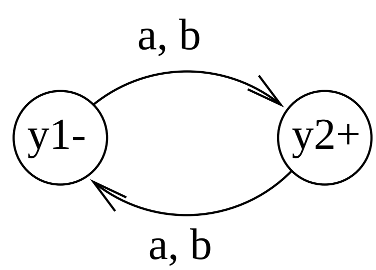
  Example: $L_1 = \{$words that end with $aa\}$
  with regular expression $\mathbf{r}_1 = (\mathbf{a}+\mathbf{b})^*\mathbf{aa}$
  $L_2 = \{$words with odd length$\}$
  with regular expression $\mathbf{r}_2 = (\mathbf{a}+\mathbf{b})((\mathbf{a}+\mathbf{b})(\mathbf{a}+\mathbf{b}))^*$

  - Consider the string $baaab$.
  - If we factor it as $(baa)(ab)$, then $baa \in L_1$ but $ab \notin L_2$.
  - However, another factorization, $(baaa)(b)$, shows that $baaab \in L_1L_2$ since $baaa \in L_1$ and $b \in L_2$.

FA1 for L1:



FA2 for L2:



- Basically idea of building FA_3 for L_1L_2 from FA_1 for L_1 and FA_2 for L_2:
  - Recall $L_1L_2 = \{w = w_1w_2 : w_1 \in L_1, w_2 \in L_2\}$.
  - So a string w is in L_1L_2 if and only if we can factor $w = w_1w_2$ such that w_1 is accepted by FA_1 and w_2 is accepted by FA_2.
  - FA_3 initially acts like FA_1.
  - When FA_3 hits a $\oplus$ state of FA_1,
    * Start a version of FA_2.
    * Keep processing on FA_1 and any previous versions of FA_2.
  - We need to keep processing on FA_1 because we don't know where the first factor w_1 ends and the second factor w_2 begins
  - Final states of FA_3 are those states that have at least one final state from FA_2.
- More formally, we build machine FA_3 in following way:
  - Suppose that FA_1 and FA_2 have the same alphabet Σ.
  - Let L_1 be language generated by regular expression $\mathbf{r}_1$ and having FA $FA_1 = (K_1, \Sigma, \pi_1, s_1, F_1)$.
  - Let L_2 be language generated by regular expression $\mathbf{r}_2$ and having FA $FA_2 = (K_2, \Sigma, \pi_2, s_2, F_2)$.
  - **Definition:** For any set S, define 2^S to be the set of all possible subsets of S.

    Example: If $S = \{a, bb, ab\}$, then

    $$2^S = \{\emptyset, \{a\}, \{bb\}, \{ab\}, \{a, bb\}, \{a, ab\}, \{bb, ab\}, \{a, bb, ab\}\}.$$

    Fact: If $|S| < \infty$, then $|2^S| = 2^{|S|}$; i.e., there are $2^{|S|}$ different subsets of S.

  - Machine $FA_3 = (K_3, \Sigma, \pi_3, s_3, F_3)$ for L_1L_2 is as follows:
    * States
      $$K_3 = \{\{x\} + Y : x \in K_1, Y \in 2^{K_2}\};$$
      i.e., each state of FA_3 is a set of states, where exactly one of the states is from FA_1 and the rest (possibly none) are from FA_2.
    * Initial state $s_3 = \{s_1\}$; i.e., the initial state of FA_3 is the set consisting of only the initial state of FA_1.

* Transition function $\pi_3 : K_3 \times \Sigma \to K_3$ is defined as

$$\begin{aligned}
&\pi_3(\{x, y_1, \ldots, y_n\}, \ell) \\
&\quad = \begin{cases} \{\pi_1(x,\ell), \pi_2(y_1,\ell), \ldots, \pi_n(y_2,\ell)\} & \text{if } \pi_1(x,\ell) \notin F_1, \\ \{\pi_1(x,\ell), \pi_2(y_1,\ell), \ldots, \pi_n(y_2,\ell), s_2\} & \text{if } \pi_1(x,\ell) \in F_1, \end{cases}
\end{aligned}$$

where $\{x, y_1, \ldots, y_n\} \in K_3$, $n \geq 0$, $x \in K_1$, $y_i \in K_2$ for $i = 1, \ldots, n$, and $\ell \in \Sigma$.

* Final states

$$F_3 = \{\{x, y_1, \ldots, y_n\} : n \geq 1, y_i \in F_2 \text{ for some } i = 1, \ldots, n\}.$$

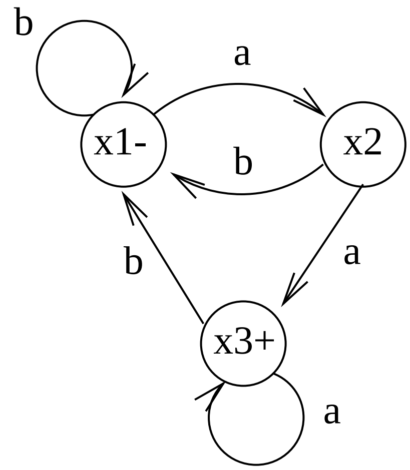
- The number of states in FA_3 is

$$|K_3| = |K_1| \cdot |2^{K_2}| = |K_1| \cdot 2^{|K_2|}.$$

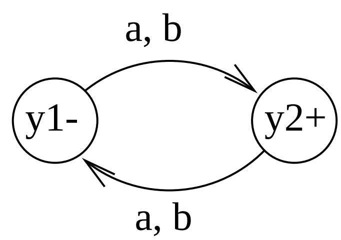
  * Actually, we can leave out from K_3 any states $\{x, y_1, \ldots, y_n\}$ that are not reachable from the initial state s_3.
  * In this case, $|K_1| \cdot 2^{|K_2|}$ still provides an upper bound for $|K_3|$; i.e., $|K_3| \leq |K_1| \cdot 2^{|K_2|}$.

Example: $L_1 = \{\text{words that end with } aa\}$
with regular expression $\mathbf{r}_1 = (\mathbf{a} + \mathbf{b})^*\mathbf{aa}$
$L_2 = \{\text{words with odd length}\}$
with regular expression $\mathbf{r}_2 = (\mathbf{a} + \mathbf{b})((\mathbf{a} + \mathbf{b})(\mathbf{a} + \mathbf{b}))^*$

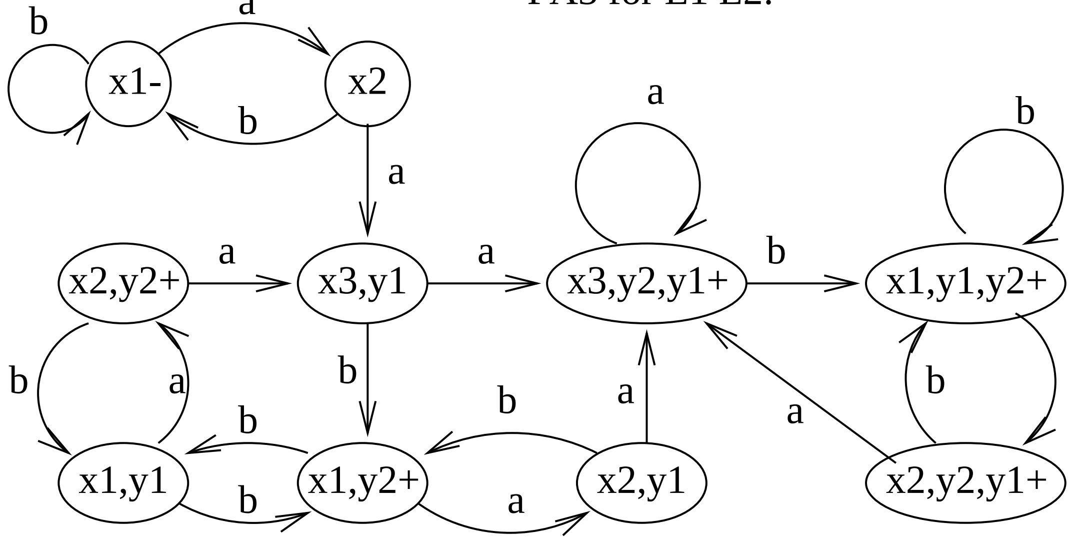
FA1 for L1:



FA2 for L2:



FA3 for L1 L2:



Rule 4: If there is an FA called FA_1 that accepts the language defined by the regular expression $\mathbf{r}_1$, then there is an FA called FA_2 that accepts the language defined by the regular expression $\mathbf{r}_1^*$.

Basic idea of how to build machine FA_2:

- Each state of FA_2 corresponds to one or more states of FA_1.
- FA_2 initially acts like FA_1.
- when FA_2 hits a $\oplus$ state of FA_1, then FA_2 simultaneously keeps track of how the rest of the string would be processed on FA_1 from where it left off and how the rest of the string would be processed on FA_1 starting in the start state.
- Whenever FA_2 hits a $\oplus$ state of FA_1, we have to start a new process starting in the start state of FA_1 (if no version of FA_1 is currently in its start state.)
- The final states of FA_2 are those states which have a correspondence to some final state of FA_1.
- We need to be careful about making sure that FA_2 accepts Λ.
- To have FA_2 accept Λ, we make the start state of FA_2 also a final state.
- But we need to be careful when there are arcs going into the start state of FA_1.

Formally, we build the machine FA_2 for L_1^* as follows:

- Let L_1 be language generated by regular expression $\mathbf{r}_1$ and having finite automaton $FA_1 = (K_1, \Sigma, \pi_1, s_1, F_1)$.
- For now, assume that FA_1 does not have any arcs entering the initial state s_1.
- Know that language L_1^* is generated by regular expression $\mathbf{r}_1^*$.
- Define $FA_2 = (K_2, \Sigma, \pi_2, s_2, F_2)$ for L_1^* with
  - States $K_2 = 2^{K_1}$.
  - Initial state $s_2 = \{s_1\}$.
  - Transition function $\pi_2 : K_2 \times \Sigma \rightarrow K_2$ with

    $$\pi_2(\{x_1, \ldots, x_n\}, \ell) \\ = \begin{cases} \{\pi_1(x_1, \ell), \ldots, \pi_1(x_n, \ell)\} & \text{if } \pi_1(x_k, \ell) \notin F_1 \text{ for all } k = 1, \ldots, n, \\ \{\pi_1(x_1, \ell), \ldots, \pi_1(x_n, \ell), s_1\} & \text{if } \pi_1(x_k, \ell) \in F_1 \text{ for some } k = 1, \ldots, n, \end{cases}$$

    where $\{x_1, \ldots, x_n\} \in K_2$, $n \geq 1$, $x_i \in K_1$ for all $i = 1, \ldots, n$, and $\ell \in \Sigma$.
  - Final states

    $$F_2 = \{s_1\} + \{\{x_1, \ldots, x_n\} : n \geq 1, x_i \in F_1 \text{ for some } i = 1, \ldots, n\}.$$

- The number of states in FA_2 is

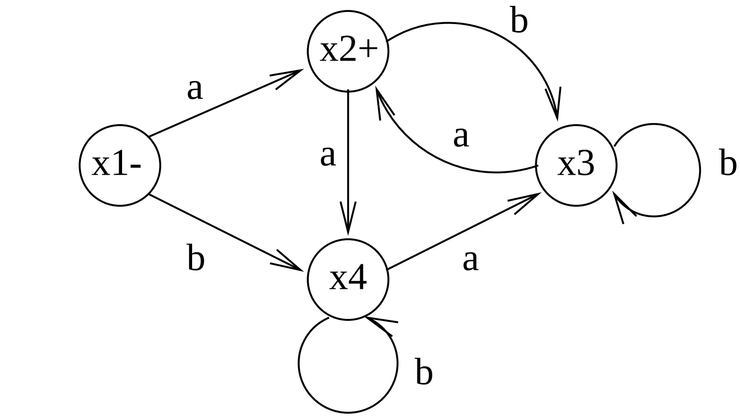
  $$|K_2| = |2^{K_1}| = 2^{|K_1|}.$$

  - Actually, we can leave out from K_2 any state $\{x_1, \ldots, x_n\}$ that is not reachable from the initial state s_2.
  - In this case, $2^{|K_1|}$ still provides an upper bound for $|K_2|$; i.e., $|K_3| \leq 2^{|K_1|}$.

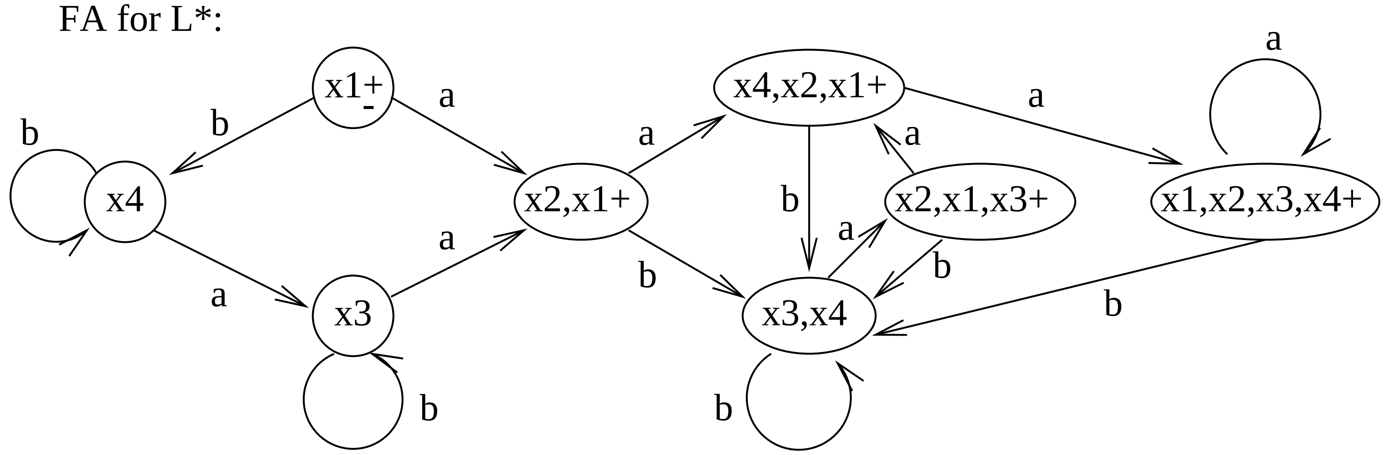
Example: Consider language L having regular expression

$$\mathbf{r} = (\mathbf{a} + \mathbf{bb}^*\mathbf{ab}^*\mathbf{a})((\mathbf{b} + \mathbf{ab}^*\mathbf{a})\mathbf{b}^*\mathbf{a})^*$$

FA for L:



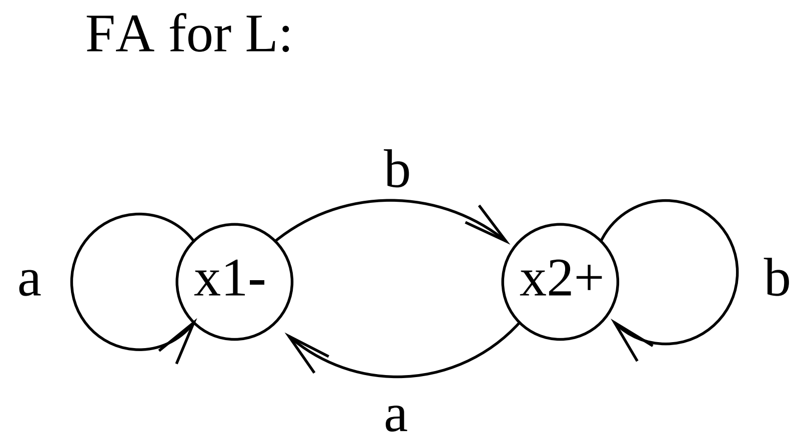
FA for L*:



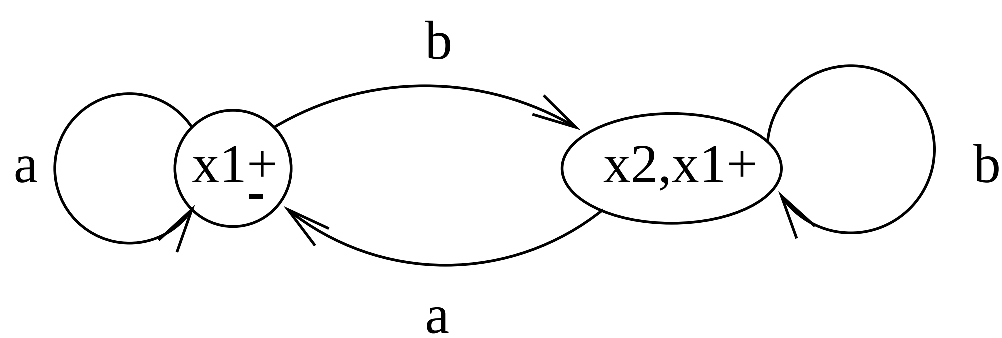
Example: Consider language L having regular expression

$$(\mathbf{a} + \mathbf{b})^*\mathbf{b}$$

Need to be careful since we can return to the start state.



If we blindly applied previous method for constructing FA for L^*, we get the following:

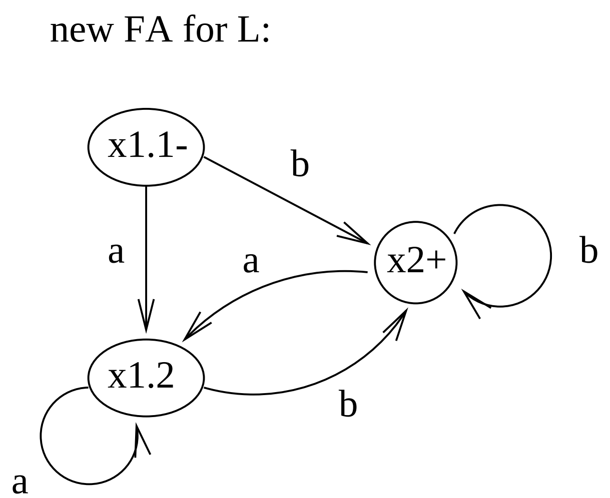


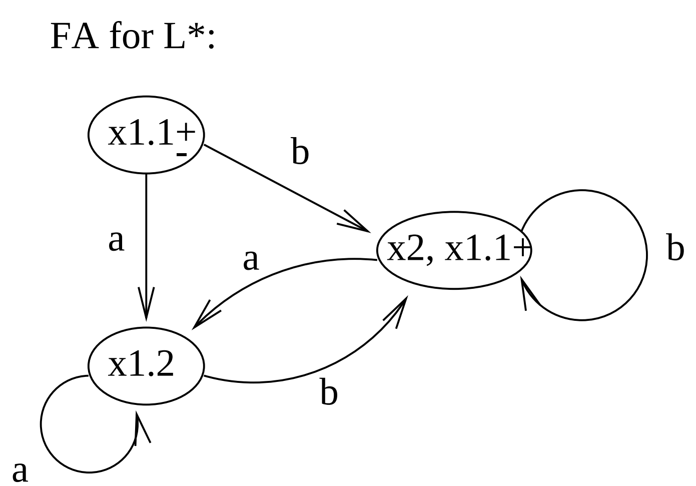
Problem:

- Note that start state is final state.
- But this FA accepts $a \notin L^*$, and so this FA is incorrect.
- Problem occurs because we can return to start state in original FA, and since we make the start state a final state in new FA.

Solution:

- Given original FA FA_1 having arcs going into the initial state, create an equivalent FA $\widetilde{FA}_1$ having no arcs going into the initial state by splitting the original start state x_1 of FA_1 into two states $x_{1.1}$ and $x_{1.2}$
  - $x_{1.1}$ is the new start state of $\widetilde{FA}_1$ and is never visited again after the first letter of the input string is read.
  - $x_{1.2}$ in $\widetilde{FA}_1$ corresponds to x_1 after the first letter of the input string is read.
- Then run algorithm to create FA for L^* from the new FA $\widetilde{FA}_1$.



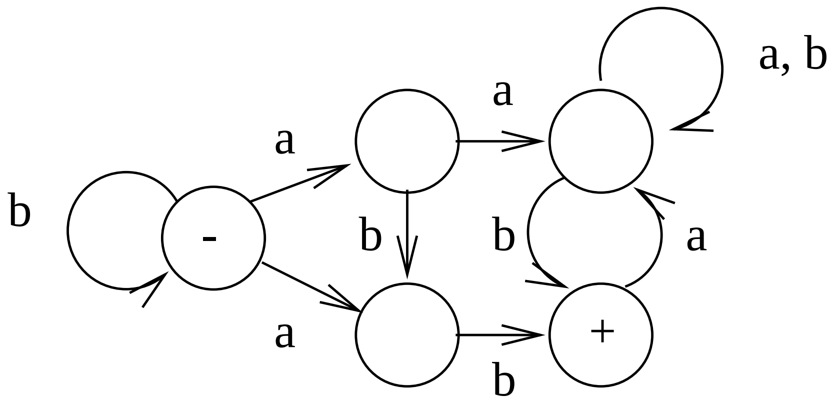


7.5 Nondeterministic Finite Automata

Definition: A *nondeterministic finite automaton* (NFA) is given by $M = (K, \Sigma, \Pi, s, F)$, where

1. K is a finite set of states.
   - $s \in K$ is the initial state, which is denoted pictorially by $\ominus$, and there is exactly one initial state.
   - $F \subset K$ is a set of final states (possibly empty), where each final state is denoted pictorially by $\bigoplus$.
2. An alphabet Σ of possible input letters.
3. $\Pi \subset K \times \Sigma \times K$ is a finite set of transitions, where each transition (arc) from one state to another state is labeled with a letter $\ell \in \Sigma$. (We do not allow for Λ to be the label of an arc since Λ is a string and not a letter of Σ.) We allow for the possibility of more than one edge with the same label from any state and there may be a state (or states) for which certain input letters have no edge leaving that state.

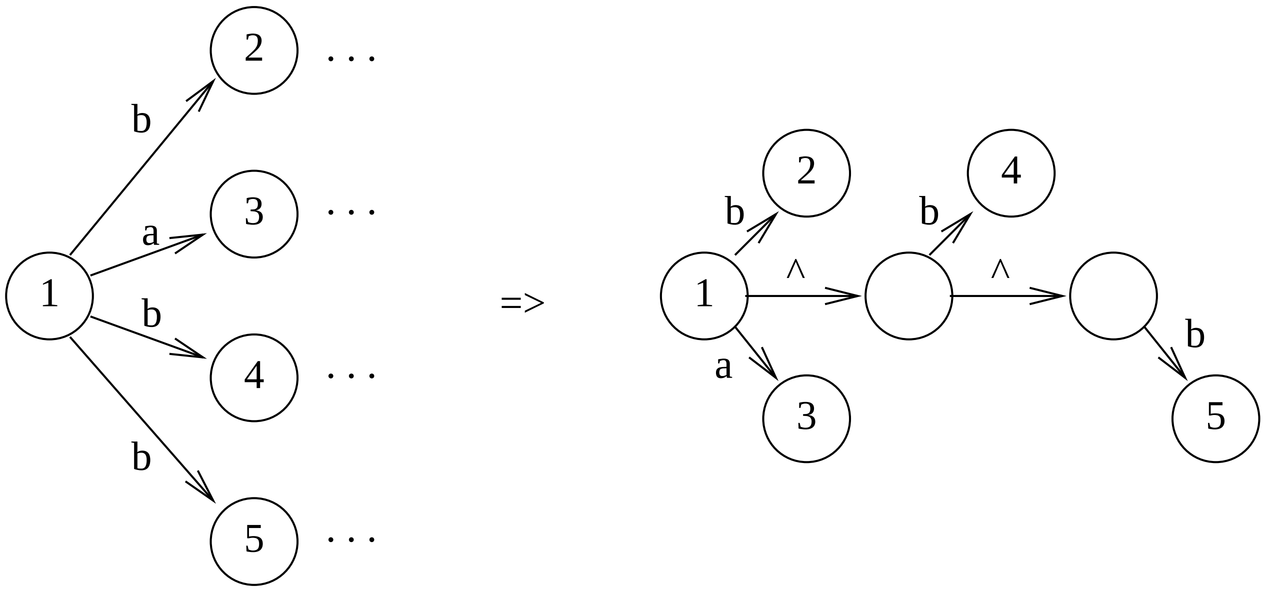
Example:



Note that

- definition of NFA is different from that of FA since
  - a FA must have from each state an arc labeled with each letter of alphabet, while NFA does not.
  - a FA is deterministic, while a NFA may be nondetermisic.
  - An NFA can have repeated labels from any single state.
- NFA allows for human choice to become a factor in selecting a way to process an input string.
- The definition of NFA is different from that of TG since
  - a TG can have arcs labeled with substrings of letters while a NFA has arcs labeled with only letters.
  - a TG can have arcs labeled with Λ while a NFA cannot.
  - a TG can have more than one start state while a NFA can only have one.

- Can transform any NFA with repeated labels from any single state to an equivalent TG with no repeated labels from any single state.



7.6 Properties of NFA

Theorem 7 *FA = NFA; i.e., any language definable by a NFA is also definable by a deterministic FA and vice versa.*

Proof. Note that

- Every FA is an NFA since we can consider an FA to be an NFA without the extra possible features.
- Every NFA is a TG.
- Kleene's theorem states that every TG has an equivalent FA.

∎

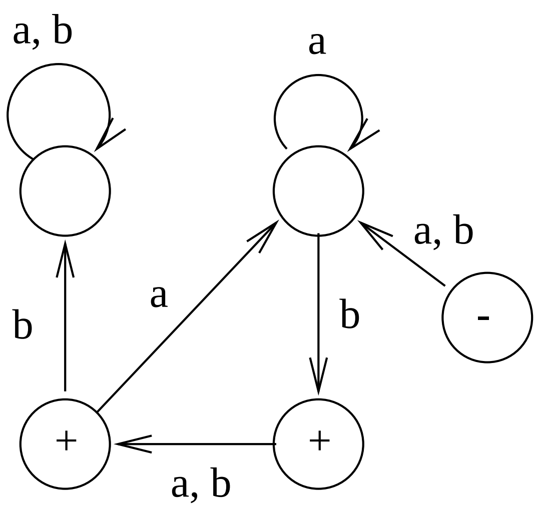
NFA useful because

- applications in artificial intelligence (AI).

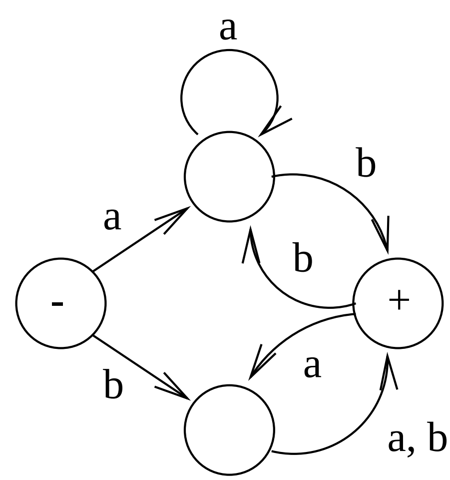
- given two FA's for two languages with regular expressions $\mathbf{r}_1$ and $\mathbf{r}_2$, it is easy to construct an NFA to accept language corresponding to regular expression $\mathbf{r}_1 + \mathbf{r}_2$.

Example:

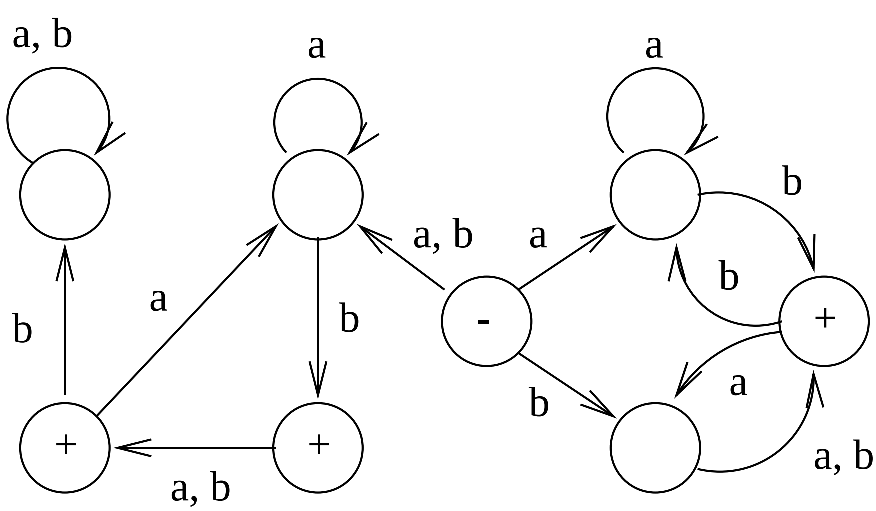
FA1:



FA2:



NFA$_{1+2}$



- This works when neither of the original FA's has any arcs going into their original initial states.
- If one or both of the original FA's has an arc going into its original initial state, the newly constructed FA for the language corresponding

to regular expression $\mathbf{r}_1 + \mathbf{r}_2$ may be incorrect. This is because the new FA may process part of the word on one of the original FA's and then process the rest of the word on the other FA, and then incorrectly accept the word.