Chapter 3

Recursive Definitions

3.1 Definition

A *recursive definition* is characteristically a three-step process:

1. First, we specify some basic objects in the set. The number of basic objects specified must be finite.
2. Second, we give a finite number of rules for constructing more objects in the set from the ones we already know.
3. Third, we declare that no objects except those constructed in this way are allowed in the set.

3.2 Examples

Example: Consider the set P-EVEN, which is the set of positive even numbers.

We can define the set P-EVEN in several different ways:

- We can define P-EVEN to be the set of all positive integers that are evenly divisible by 2.
- P-EVEN is the set of all $2n$, where $n = 1, 2, \ldots$.

- P-EVEN is defined by these three rules:

  Rule 1 2 is in P-EVEN.

  Rule 2 If x is in P-EVEN, then so is $x + 2$.

  Rule 3 The only elements in the set P-EVEN are those that can be produced from the two rules above.

Note that the first two definitions of P-EVEN are much easier to apply than the last.

In particular, to show that 12 is in P-EVEN using the last definition, we would have to do the following:

1. 2 is in P-EVEN by Rule 1.
2. $2 + 2 = 4$ is in P-EVEN by Rule 2.
3. $4 + 2 = 6$ is in P-EVEN by Rule 2.
4. $6 + 2 = 8$ is in P-EVEN by Rule 2.
5. $8 + 2 = 10$ is in P-EVEN by Rule 2.
6. $10 + 2 = 12$ is in P-EVEN by Rule 2.

We can make another definition for P-EVEN as follows:

Rule 1 2 is in P-EVEN.

Rule 2 If x and y are both in P-EVEN, then $x + y$ is in P-EVEN.

Rule 3 No number is in P-EVEN unless it can be produced by rules 1 and 2.

Can use the new definition of P-EVEN to show that 12 is in P-EVEN:

1. 2 is in P-EVEN by Rule 1.
2. $2 + 2 = 4$ is in P-EVEN by Rule 2.
3. $4 + 4 = 8$ is in P-EVEN by Rule 2.
4. $4 + 8 = 12$ is in P-EVEN by Rule 2.

Example: Let PALINDROME be the set of all strings over the alphabet $\Sigma = \{a, b\}$ that are the same spelled forward as backwards; i.e., PALINDROME $= \{w : w = \text{reverse}(w)\} = \{\Lambda, a, b, aa, bb, aaa, aba, bab, bbb, aaaa, abba, \ldots\}$.

A recursive definition for PALINDROME is as follows:

Rule 1 Λ, a, and b are in PALINDROME.

Rule 2 If $w \in$ PALINDROME, then so are awa and bwb.

Rule 3 No other string is in PALINDROME unless it can be produced by rules 1 and 2.

Example: Let us now define a set AE of certain valid arithmetic expressions. The set AE will not include all possible arithmetic expressions.

The alphabet of AE is

$$\Sigma = \{0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ +\ -\ *\ /\ (\ )\}$$

We recursively define AE using the following rules:

Rule 1 Any number (positive, negative, or zero) is in AE.

Rule 2 If x is in AE, then so are (x) and $-(x)$.

Rule 3 If x and y are in AE, then so are

- **(i)** $x + y$ (if the first symbol in y is not $-$)
- **(ii)** $x - y$ (if the first symbol in y is not $-$)
- **(iii)** $x * y$
- **(iv)** x/y
- **(v)** $x * *y$ (our notation for exponentiation)

Rule 4 AE consists of only those things can be created by the above three rules.

For example,

$$(5 * (8 + 2))$$

and

$$5 - (8 + 1)/3$$

are in AE since they can be generated using the above definition.

However,

$$((6 + 7)/9$$

and

$$4(/9 * 4)$$

are not since they cannot be generated using the above definition.

Now we can use our recursive definition of AE to show that

$$8 * 6 - ((4/2) + (3 - 1) * 7)/4$$

is in AE.

1. Each of the numbers are in AE by Rule 1.
2. $8 * 6$ is in AE by Rule 3(iii).
3. $4/2$ is in AE by Rule 3(iv).
4. $(4/2)$ is in AE by Rule 2.
5. $3 - 1$ is in AE by Rule 3(ii).
6. $(3 - 1)$ is in AE by Rule 2.
7. $(3 - 1) * 7$ is in AE by Rule 3(iii).
8. $(4/2) + (3 - 1) * 7$ is in AE by Rule 3(i).
9. $((4/2) + (3 - 1) * 7)$ is in AE by Rule 2.
10. $((4/2) + (3 - 1) * 7)/4$ is in AE by Rule 3(iv).
11. $8 * 6 + ((4/2) + (3 - 1) * 7)/4$ is in AE by Rule 3(i).