Chapter 1

Introduction

1.1 Purpose of Course

Course covers the theory of computers:

- Not concerned with actual hardware and software.
- More interested in abstract questions of the frontiers of capability of computers.
- More specifically, what can and what cannot be done by any existing computer or any computer ever built in the future.
- We will study different types of theoretical machines that are mathematical models for actual physical processes.
- By considering the possible inputs on which these machines can work, we can analyze their various strengths and weaknesses.
- We can then develop what we may believe to be the most powerful machine possible.
- Surprisingly, it will not be able to perform every task, even some easily described tasks.

1.2 Mathematical Background

- In this class, we will be seeing a number of theorems and proofs.
- To be able to understand how to prove a theorem, we first have to understand how theorems are stated.
- Many (but not all) theorems are stated as "if p, then q", where p and q are statements.

Example: If a word w has more e's than o's, then w has at least one e.

Example: If a word w has m a's and n e's in it, then the word w has at least $m + n$ letters in all.

Example: If $x^2 = 0$, then $x = 0$.

So what does "if p, then q" mean?

- If a theorem stated in this form is to be true, then it means that if p is true, then q must also be true.
- Note that this does not say that if q is true, then p must also be true. This may or may not be the case.

Example: The statement, "If a word w has at least one e, then w has more e's than o's" is not true.

For example, consider the word "exploration" or "Exxon."

Example: If a word w has at least $m + n$ letters in all, then the word w has m a's and n e's in it.

For example, suppose $m = n = 1$, and consider the word "goof."

Example: If $x = 0$, then $x^2 = 0$.

So now how do we prove a result?

We do it by arguing very carefully, where each step in our argument follows logically from the previous step.

There are several ways of proving that a statement "if p, then q" holds:

- One way is to use a direct argument:

  Example: Prove: If a word w has more e's than o's, then w has at least one e.

  Proof. Let n_e be the number of e's in w, and let n_o be the number of o's in w. Since w has more e's than o's, we must have that $n_e > n_o$, or in other words $n_e \geq n_o + 1$. But since w cannot have fewer than zero o's, we must have that $n_o \geq 0$. Therefore, $n_e \geq n_o + 1 \geq 0 + 1 = 1$. Thus, w has at least one e. ∎

- Another way of proving results is by contradiction. We do this by assuming that p is true and that q is not true, and then showing that an inconsistency results.

  Example: Prove: If $x^2 = 0$, then $x = 0$.

  Proof. Suppose that $x^2 = 0$ but $x \neq 0$. Then either $x > 0$ or $x < 0$. But if $x > 0$, then $x^2 > 0$, and if $x < 0$, then $x^2 > 0$. In either case, $x^2 > 0$. This contradicts the assumption that $x^2 = 0$. ∎

  Example: Prove: If $x > 0$ with $x \in \Re$, then $x^2 > 0$.

  Proof. Suppose that $x^2 = 0$. Then $x = 0$ so $x \not> 0$. ∎

There are several equivalent ways of stating "if p, then q"

- "if not q, then not p"
- "p only if q"
- "q if p"
- "p implies q"
- "p is sufficient for q"
- "q is necessary for p"

Example: Let x be a real number. If $x > 0$, then $x^2 > 0$.

This is equivalent to stating

- "If $x^2 > 0$ is not true (i.e., $x^2 \leq 0$), then $x > 0$ is not true (i.e., $x \leq 0$)."
- This is also equivalent to stating "$x > 0$ only if $x^2 > 0$."
- This is also equivalent to stating "$x^2 > 0$ if $x > 0$."
- This is also equivalent to stating "$x > 0$ implies $x^2 > 0$."

Often, the two statements

1. "p only if q" (i.e., "if p, then q") and
2. "p if q" (i.e., "if q, then p")

are combined into "p if and only if q" (or "p is a necessary and sufficient condition for q").

In order for this statement to be true, we need to show that both statements 1 and 2 above are true.

Definition: An integer n is an *even number* if $n = 2k$ for some $k = 0, 1, 2, 3, \ldots$.

Definition: An integer n is an *odd number* if $n = 2k + 1$ for some $k = 0, 1, 2, 3, \ldots$.

Definition: An integer n is a *positive even number* if $n = 2k$ for some $k = 1, 2, 3, \ldots$.